MSO 201A / ESO 209: Probability and Statistics
2014-2015-II Semester
End Semester Examination

Note: (i) Start the answer of every question in a fresh page and do all parts of question at one place. Questions whose different parts are done at different places may not be graded.
(ii) Write your answers legibly and present your arguments clearly and concisely. Box your final answers and conclusions.

1 Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the distribution function of random vector (X, Y). Suppose that $F(x, y) = \min\{x, y\}$, $0 \le x, y \le 1$.
(i) Define $F(x, y)$ in other regions of $\mathbb{R}^2$;
(ii) Find marginal distribution function of X;
(iii) Find $P(\frac{1}{4} < X \le \frac{3}{4}, Y > \frac{1}{2})$;
(iv) Find $P(X = Y)$. $\boxed{2+2+3+3=10\text{Marks}}$

2 Let Z_1 and Z_2 be i.i.d. $N(0, 1)$ random variables and let $Y_1 = Z_1 + Z_2$ and $Y_2 = Z_1^2 + Z_2^2$.
(i) Find the joint m.g.f. of $\underline{Y} = (Y_1, Y_2)$;
(ii) Find the correlation coefficient between Y_1 and Y_2;
(iii) Are Y_1 and Y_2 independently distributed?; $\boxed{4+4+2=10\text{Marks}}$

3 The unit interval $(0, 1)$ is split into three parts by picking two points (say, U_1 and U_2) at random from the interval (i.e., U_1 and U_2 are i.i.d. $U(0, 1)$ random variables). Let Y_1, Y_2 and Y_3 be the lengths of three parts so obtained. Show that Y_1, Y_2 and Y_3 are identically distributed. **(Hint:** $Y_1 = \min\{U_1, U_2\}, Y_2 = \max\{U_1, U_2\} - \min\{U_1, U_2\}$ and $Y_3 = 1 - \max\{U_1, U_2\}$) $\boxed{10\text{Marks}}$

4 Let Z_1 and Z_2 be i.i.d. random variables with common p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y_1 = Z_1 + |Z_2|$ and $Y_2 = Z_2$.
(i) Using the transformation of variable technique (Jacobian method) find the joint p.d.f. of $\underline{Y} = (Y_1, Y_2)$;
(ii) Find the marginal p.d.f. of Y_1. $\boxed{6+4=10\text{Marks}}$

5 (i) Let $1 \le r_1 \le r_2$ be positive integers and let $x \in (0, 1)$. Show that

$$\frac{1}{B(r_1, r_2 - r_1 + 1)} \int_0^x t^{r_1 - 1}(1 - t)^{r_2 - r_1} dt = \sum_{j=r_1}^{r_2} \binom{r_2}{j} x^j (1 - x)^{r_2 - j}.$$

(ii) Let $U \sim U(0, 1)$ and let X be the root of the equation

$$\sum_{j=1}^{5} \binom{5}{j} X^j (1 - X)^{5-j} = U.$$

Find $P(X > \frac{1}{2})$;
(iii) Let X_1 and X_2 be i.i.d. Be$(3, 5)$ random variables and let

$$Y_i = 210 \int_0^{X_i} t^2 (1 - t)^4 dt, \ i = 1, 2.$$

Find $P(Y_1 + Y_2 \leq 3)$. $\boxed{4+4+2=10\text{Marks}}$

6 Let $\underline{X} = (X_1, X_2, X_3)$ have the joint p.m.f.

$$f(x_1, x_2, x_3) = \begin{cases} cx_1x_2x_3, & \text{if } x_1, x_2, x_3 \in \{1, 2, 3\},\ x_1 \leq x_2 \leq x_3 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Find the constant c;

(ii) Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the conditional variance of Y_1 given $Y_2 = 5$;

(iii) Let $Y_3 = X_2 - X_1$ and $Y_4 = X_3 - X_2$. Are Y_3 and Y_4 independently distributed random variables? $\boxed{3+4+3=10\text{Marks}}$

7 (i) Let $X_n \sim \text{Bin}(n, p_n)$, $n = 1, 2, \ldots$, where $p_n \in (0, 1)$, $n = 1, 2, \ldots$ and $\lim_{n\to\infty}(np_n) = \lambda$, for some $\lambda > 0$. Show that

$$\lim_{n\to\infty} f_{X_n}(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases},$$

where $f_{X_n}(x)$ is the p.m.f. of $X_n, n = 1, 2, \ldots$.

(ii) A person plays a series of 5000 games independently and the probability of person winning any game is 0.001. Find the approximate probability that he will win at least 3 games. $\boxed{6+4=10\text{Marks}}$

8 Let $X_1, X_2, \ldots Z_n$ be a collection of i.i.d. $N(\mu, 1)$ random variables, where $-\infty < \mu < \infty$. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = 1$. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $Y = \sum_{i=1}^n a_i X_i$.

(i) Show that $\overline{X}$ and Y are independently distributed;

(ii) Find the p.d.f. of $Y_1 = \frac{\sqrt{n}\overline{X}}{|Y|}$;

(iii) Find the p.d.f. of $Y_2 = \frac{|X_1 - X_2|}{|X_3 + X_4|}$. $\boxed{3+4+3=10\text{Marks}}$

9 (i) Let $X_1, \ldots, X_n$ be random variables and let $p_1, \ldots, p_n$ be positive real numbers such that $\sum_{i=1}^n p_i = 1$. Show that $\boxed{5+5=10 \text{ Marks}}$

$$\sqrt{\text{Var}(\sum_{i=1}^n p_i X_i)} \leq \sum_{i=1}^n p_i \sqrt{\text{Var}(X_i)} \leq \sqrt{\sum_{i=1}^n p_i \text{Var}(X_i)}.$$

(ii) Let $a_1, \ldots, a_n$ and $p_1, \ldots, p_n$ be positive real numbers such that $|a_i| > 1, i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = 1$. Show that

$$\sum_{i=1}^n a_i(a_i - 1)p_i \geq (\sum_{i=1}^n \sqrt{a_i} p_i)^2((\sum_{i=1}^n \sqrt{a_i} p_i)^2 - 1).$$

10. (i) Let Z_1 and Z_2 be i.i.d. $N(0, 1)$ random variables. Find the conditional p.d.f. of $2Z_1 + 3Z_2$ given that $Z_1 = 1$.

(ii) Let (Z_1, Z_2) have the bivariate normal distribution with $E(Z_1) = E(Z_2) = 0$, $\text{Var}(Z_1) = \text{Var}(Z_2) = 1$ and $\text{Cov}(Z_1, Z_2) = \frac{1}{2}$. Find the value of $E(\max\{Z_1, Z_2\})$. (Hint: for real numbers a and b, $\max\{a, b\} = \frac{a+b+|a-b|}{2}$). $\boxed{5+5=10 \text{ Marks}}$

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Model Solutions

Problem No. 1

(i) $$F(x,y) = \begin{cases} 0, & \text{if } x<0 \text{ or } y<0 \\ \min\{x,y\}, & \text{if } 0\le x<1,\ 0\le y<1 \\ x, & \text{if } 0\le x<1,\ y\ge 1 \\ y, & \text{if } x\ge 1,\ 0\le y<1 \\ 1, & \text{if } x\ge 1,\ y\ge 1 \end{cases}$$

.... $\boxed{\text{2 MARKS}}$

(ii) $$F_X(x) = \lim_{y\to\infty} F(x,y) = \begin{cases} 0, & \text{if } x<0 \\ x, & \text{if } 0\le x<1 \\ 1, & \text{if } x\ge 1 \end{cases}$$

.... $\boxed{\text{2 MARKS}}$

(iii) $$P\left(\tfrac{1}{4}<X\le\tfrac{3}{4},\ Y>\tfrac{1}{2}\right) = P\left(\tfrac{1}{4}<X\le\tfrac{3}{4}\right) - P\left(\tfrac{1}{4}<X\le\tfrac{3}{4},\ Y\le\tfrac{1}{2}\right)$$

$$= F_X\left(\tfrac{3}{4}\right) - F_X\left(\tfrac{1}{4}\right) - \left[F\left(\tfrac{3}{4},\tfrac{1}{2}\right) - F\left(\tfrac{1}{4},\tfrac{1}{2}\right)\right]$$

$$= \tfrac{3}{4} - \tfrac{1}{4} - \left[\tfrac{1}{2} - \tfrac{1}{4}\right] = \tfrac{1}{4}$$

..... $\boxed{\text{3 MARKS}}$

(iv) Clearly F is the d.f. of (X, X).

$\Rightarrow P(X=Y)=1.$ .... $\boxed{\text{3 MARKS}}$

$\boxed{\text{Problem No. 2}}$ (i) $M_{Y_1,Y_2}(t_1,t_2) = E\left[e^{t_1(Z_1+Z_2)+t_2(Z_1^2+Z_2^2)}\right]$

$$= E\left[e^{t_1Z_1+t_2Z_1^2}\, e^{t_1Z_2+t_2Z_2^2}\right]$$

$$= E\left(e^{t_1Z_1+t_2Z_1^2}\right) E\left(e^{t_1Z_2+t_2Z_2^2}\right) \quad [\text{Independence}]$$

$$= \left[E\left(e^{t_1Z_1+t_2Z_1^2}\right)\right]^2, \quad \underset{\sim}{t} \in \mathbb{R}^2.$$

.... $\boxed{\text{2 MARKS}}$

$$E\left(e^{t_1Z_1+t_2Z_1^2}\right) = \int_{-\infty}^{\infty} e^{t_1 z+t_2 z^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z^2-2t_2z^2-2t_1z\right]} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1-2t_2}{2}\left[z^2-\frac{2t_1}{1-2t_2}z\right]} dz$$

$$= e^{-\frac{t_1^2}{2(1-2t_2)}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1-2t_2}{2}\left(z-\frac{t_1}{1-2t_2}\right)^2} dz$$

$$= \frac{e^{-\frac{t_1^2}{2(1-2t_2)}}}{\sqrt{1-2t_2}}, \quad t_1 \in \mathbb{R},\ t_2 < \tfrac{1}{2}$$

$$\Rightarrow M_{Y_1,Y_2}(t_1,t_2) = \frac{e^{-\frac{t_1^2}{1-2t_2}}}{1-2t_2}, \quad t_1 \in \mathbb{R},\ t_2 < \tfrac{1}{2}$$

.... $\boxed{\text{2 MARKS}}$

(ii) $\psi_{\underset{\sim}{Y}}(\underset{\sim}{t}) = \ln M_{\underset{\sim}{Y}}(\underset{\sim}{t}) = -\frac{t_1^2}{1-2t_2} - \ln(1-2t_2), \quad t_2 < \frac{1}{2}$

$$\mathrm{Cov}(Y_1,Y_2) = \left[\frac{\partial^2}{\partial t_1 \partial t_2} \psi_{\underset{\sim}{Y}}(\underset{\sim}{t})\right]_{\underset{\sim}{t}=\underset{\sim}{0}} = \left[\frac{-2t_1}{(1-2t_2)^2}\right]_{\underset{\sim}{t}=\underset{\sim}{0}} = 0$$

.... $\boxed{\text{4 MARKS}}$

$\Rightarrow \rho(Y_1,Y_2) = 0$

(iii) $M_{Y_1}(t) = M_{Y_1,Y_2}(t,0) = e^{-t^2}, t \in \mathbb{R}; \quad M_{Y_2}(t) = M_{Y_1,Y_2}(0,t) = (1-2t)^{-1}, \ t < \frac{1}{2}$

$$M_{Y_1,Y_2}(t_1,t_2) = \frac{e^{-\frac{t_1^2}{1-2t_2}}}{1-2t_2} \neq M_{Y_1}(t_1)\, M_{Y_2}(t_2) \ \forall\ t_1 \in \mathbb{R},\ t_2 < \tfrac{1}{2}$$

$\Rightarrow Y_1$ and Y_2 are not independent ... $\boxed{\text{2 MARKS}}$

$\boxed{2/10}$

$\boxed{\text{Problem No. 3}}$ Let $U_{(1)} = \min\{U_1, U_2\}$ and $U_{(2)} = \max\{U_1, U_2\}$. Then

$$f_{U_{(1)}, U_{(2)}}(u_1, u_2) = \begin{cases} 2, & 0 < u_1 < u_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

We have $Y_1 = U_{(1)}$, $Y_2 = U_{(2)} - U_{(1)}$ and $Y_3 = 1 - U_{(2)}$

$$\Rightarrow U_{(1)} = Y_1, \quad U_{(2)} = Y_1 + Y_2$$

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$0 < u_1 < u_2 < 1 \iff 0 < y_1 < y_1 + y_2 < 1 \iff y_1 > 0, \ y_2 > 0, \ y_1 + y_2 < 1$$

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 2 & \text{if } y_1 > 0, \ y_2 > 0, \ y_1 + y_2 < 1 \\ 0 & \text{o.w.} \end{cases} \quad \ldots \boxed{\text{4 MARKS}}$$

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2)\, dy_2 = \begin{cases} \int_0^{1-y_1} 2\, dy_2, & \text{if } 0 < y_1 < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2(1-y_1), & 0 < y_1 < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2)\, dy_1$$

$$= \begin{cases} 2(1-y_2), & 0 < y_2 < 1 \\ 0 & \text{o.w.} \end{cases}$$

Clearly, $f_{Y_1}(y) = f_{Y_2}(y)$, $\forall y \in \mathbb{R} \Rightarrow Y_1 \stackrel{d}{=} Y_2 \ldots \boxed{\text{4 MARKS}}$

$$f_{U_{(2)}}(u_2) = \int_{-\infty}^{\infty} f_{U_{(1)}, U_{(2)}}(u_1, u_2)\, du_1 = \begin{cases} \int_0^{u_2} 2\, du_1 & 0 < u_2 < 1 \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} 2u_2, & 0 < u_2 < 1 \\ 0 & \text{o.w} \end{cases}$$

Thus pdf of $Y_3 = 1 - U_{(2)}$ is

$$f_{Y_3}(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{o.w} \end{cases} = f_{Y_1}(y) = f_{Y_2}(y), \ \forall y$$

$$\Rightarrow Y_3 \stackrel{d}{=} Y_1 \stackrel{d}{=} Y_2$$

$\ldots \boxed{\text{2 MARKS}}$

$\boxed{3/10}$

$\boxed{\text{Problem No. 4}}$ (i) $f_{Z_1, Z_2}(z_1, z_2) = \begin{cases} \frac{1}{9}, & -1 < z_1 < 2, \ -1 < z_2 < 2 \\ 0, & \text{o.w.} \end{cases}$

$S_{\underset{\sim}{Z}} = (-1, 2) \times (-1, 2)$; $\quad Y_1 = h_1(Z_1, Z_2) = |Z_1| + Z_2 \quad Y_2 = h_2(Z_1, Z_2) = Z_2$.

$\underset{\sim}{h} = (h_1, h_2) : S_{\underset{\sim}{Z}} \to \mathbb{R}$ is not 1-1

$S_1 = (-1, 0) \times (-1, 2)$

$y_1 = h_1(z_1, z_2) = -z_1 + z_2, \ y_2 = h_2(z_1, z_2) = z_2$

$h_{1,1}^{-1}(y_1, y_2) = y_2 - y_1, \quad h_{1,2}^{-1}(y_1, y_2) = y_2$

$J_1 = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$\underset{\sim}{h}(S_1) = \{(y_1, y_2) \in \mathbb{R}^2 : \max\{-1, y_1 - 1\} < y_2 < \min\{2, y_1\}, \ -1 < y_1 < 3\}$

$S_2 = (0, 1) \times (-1, 2)$

$y_1 = h_1(z_1, z_2) = z_1 + z_2, \ y_2 = h_2(z_1, z_2) = z_2$

$h_{2,1}^{-1}(y_1, y_2) = y_1 - y_2, \ h_{2,2}^{-1}(y_1, y_2) = y_2$

$J_2 = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$\underset{\sim}{h}(S_2) = \{(y_1, y_2) \in \mathbb{R}^2 : \max\{-1, y_1 - 2\} < y_2 < \min\{2, y_1\}, \ -1 < y_1 < 4\}$

.... $\boxed{3 \text{ MARKS}}$

$$f_{\underset{\sim}{Y}}(y_1, y_2) = \begin{cases} \frac{2}{9}, & \text{if } -1 < y_1 < 0, \ -1 < y_2 < y_1 \text{ or } 0 < y_1 < 1, \ y_1 - 1 < y_2 < y_1 \text{ or } \\ & 1 < y_1 < 2, \ y_1 - 1 < y_2 < y_1 \text{ or } 2 < y_1 < 3, \ y_1 - 1 < y_2 < 2 \\ \frac{1}{9}, & \text{if } 0 < y_1 < 1, \ -1 < y_2 < y_1 - 1 \text{ or } 1 < y_1 < 2, \ y_1 - 2 < y_2 < y_1 - 1 \text{ or } \\ & 2 < y_1 < 3, \ y_1 - 2 < y_2 < y_1 - 1 \text{ or } 3 < y_1 < 4, \ y_1 - 2 < y_2 < 2 \\ 0, & \text{o.w.} \end{cases}$$

... $\boxed{3 \text{ MARKS}}$

(ii) $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{\underset{\sim}{Y}}(y_1, y_2)\, dy_2$

$$= \begin{cases} \frac{2(y_1+1)}{9}, & \text{if } -1 < y_1 < 0 \\ \frac{y_1+2}{9}, & \text{if } 0 < y_1 < 1 \\ \frac{3}{9}, & \text{if } 1 < y_1 < 2 \\ \frac{7-2y_1}{9}, & \text{if } 2 < y_1 < 3 \\ \frac{4-y_1}{9}, & \text{if } 3 < y_1 < 4 \\ 0, & \text{otherwise} \end{cases}$$

... $\boxed{4 \text{ MARKS}}$

$\boxed{4/10}$

Problem No. 5 (i) LHS $= I_{r_1,r_2}(\lambda) = \dfrac{\lfloor r_2}{\lfloor r_1-1 \; \lfloor r_2-r_1} \displaystyle\int_0^\lambda t^{r_1-1}(1-t)^{r_2-r_1}\,dt$

$$= \frac{\lfloor r_2}{\lfloor r_1-1 \; \lfloor r_2-r_1}\left[\left\{ \frac{t^{r_1}}{r_1}(1-t)^{r_2-r_1} \right\}_0^\lambda + \int_0^\lambda \frac{t^{r_1}}{r_1}(r_2-r_1)(1-t)^{r_2-r_1-1}\,dt \right]$$

$$= \binom{r_2}{r_1}\lambda^{r_1}(1-\lambda)^{r_2-r_1} + \frac{\lfloor r_2}{\lfloor r_1 \; \lfloor r_2-r_1-1}\int_0^\lambda t^{r_1}(1-t)^{r_2-r_1-1}\,dt$$

.... $\boxed{2 \text{ MARKS}}$

$$= \binom{r_2}{r_1}\lambda^{r_1}(1-\lambda)^{r_2-r_1} + I_{r_1+1,r_2}(\lambda)$$

$$= \binom{r_2}{r_1}\lambda^{r_1}(1-\lambda)^{r_2-r_1} + \binom{r_2}{r_1+1}\lambda^{r_1+1}(1-\lambda)^{r_2-r_1-1} + I_{r_1+2,r_2}(\lambda)$$

$$= \binom{r_2}{r_1}\lambda^{r_1}(1-\lambda)^{r_2-1} + \binom{r_2}{r_1+1}\lambda^{r_1+1}(1-\lambda)^{r_2-r_1-1} + \cdots \binom{r_2}{r_2-1}\lambda^{r_2-1}(1-\lambda)$$

$$+ \underbrace{I_{r_2,r_2}(\lambda)}_{= r_2\int_0^\lambda t^{r_2-1}\,dt}$$

$$= \sum_{j=r_1}^{r_2}\binom{r_2}{j}\lambda^j(1-\lambda)^{r_2-j}$$

.... $\boxed{2 \text{ MARKS}}$

(ii) Using (i) we have

$$\sum_{j=1}^{5}\binom{5}{j}\lambda^j(1-\lambda)^{5-j} = G(\lambda),$$

where G is d.f. $Be(1,5)$.

$G(X) = U \Rightarrow X = G^{-1}(U) \sim Be(1,5)$ .... $\boxed{2 \text{ MARKS}}$

$$\Rightarrow P(X > \tfrac{1}{2}) = \frac{1}{B(1,5)}\int_{1/2}^{1}(1-t)^4\,dt = \frac{\lfloor 5}{\lfloor 4}\int_0^{1/2} t^4\,dt = \frac{1}{32}$$

.... $\boxed{2 \text{ MARKS}}$

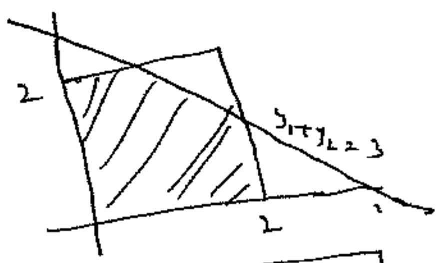
(iii) The common d.f. of X_1 and X_2 is

$$F(\lambda) = \frac{1}{B(3,5)}\int_0^\lambda t^2(1-t)^4\,dt = 105\int_0^\lambda t^2(1-t)^4\,dt$$

$\Rightarrow F(X_1)$ and $F(X_2)$ are i.i.d $U(0,1)$

$\Rightarrow \frac{Y_1}{2}$ and $\frac{Y_2}{2}$ are iid $U(0,1)$

$\Rightarrow Y_1$ and Y_2 are iid $U(0,2)$



$P(Y_1+Y_2 \le 3) = \frac{1}{4} \times$ Area of shaded region $= \frac{1}{4}\left[4 - \frac{1}{2}\right] = 7/8$ ... $\boxed{2 \text{ MARKS}}$

$\boxed{5/10}$

$\boxed{\text{Problem No. 6}}$ $S_{\underline{x}} = \{(1,1,1), (1,1,2), (1,1,3), (1,2,2), (1,2,3), (1,3,3), (2,2,2), (2,2,3), (2,3,3), (3,3,3)\}$

(i) $\sum_{\underline{x} \in S_{\underline{x}}} f(\underline{x}) = 1 \Rightarrow c\,[1+2+3+4+6+9+8+12+18+27] = 1$

$\Rightarrow c = \frac{1}{90}$ ..... $\boxed{\text{3MARKS}}$

(ii) $Y_1 = X_1 + X_2$, $Y_2 = X_2 + X_3$

$S_{\underline{y}} = \{(2,2), (2,3), (2,4), (3,4), (3,5), (4,6), (4,4), (4,5), (5,6), (6,6)\}$

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} \frac{1}{90}, & \underline{y} = (2,2) \\ \frac{2}{90}, & \underline{y} = (2,3) \\ \frac{3}{90}, & \underline{y} = (2,4) \\ \frac{4}{90}, & \underline{y} = (3,4) \\ \frac{6}{90}, & \underline{y} = (3,5) \\ \frac{9}{90}, & \underline{y} = (4,6) \\ \frac{8}{90}, & \underline{y} = (4,4) \\ \frac{12}{90}, & \underline{y} = (4,5) \\ \frac{18}{90}, & \underline{y} = (5,6) \\ \frac{27}{90}, & \underline{y} = (6,6) \\ 0, & \text{o.w.} \end{cases}$$

$P(Y_2 = 5) = \frac{6}{90} + \frac{12}{90} = \frac{1}{5}$

$f_{Y_1|Y_2}(y|5) = \frac{P(Y_1 = y, Y_2 = 5)}{P(Y_2 = 5)}$

$$= \begin{cases} \frac{1}{3}, & y = 3 \\ \frac{2}{3}, & y = 4 \\ 0, & \text{o.w.} \end{cases}$$

$E(Y_1|Y_2 = 5) = 1 + \frac{8}{3} = \frac{11}{3}$ .. $\boxed{\text{2MARKS}}$

$E(Y_1^2|Y_2 = 5) = 3 + \frac{32}{3} = \frac{41}{3}$

$Var(Y_1|Y_2 = 5) = E(Y_1^2|Y_2 = 5) - (E(Y_1|Y_2 = 5))^2$

$= \frac{2}{9}$ .... $\boxed{\text{2MARKS}}$

(iii) $Y_3 = X_2 - X_1$, $Y_4 = X_3 - X_2$; $S_{\underline{y}} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$

$P(Y_3 = 0, Y_4 = 0) = P(X_1 = X_2 = X_3) = \frac{1}{90}[1 + 8 + 27] = \frac{36}{90}$

$P[Y_3 = 0] = P[X_1 = X_2] = \frac{1}{90}(1+2+3+8+12+27) = \frac{53}{90}$ ... $\boxed{\text{3MARKS}}$

$P(Y_4 = 0) = P[X_2 = X_3] = \frac{1}{90}[1+4+9+8+18+27] = \frac{67}{90}$

$P(Y_3 = 0, Y_4 = 0) \neq P(Y_3 = 0)\, P(Y_4 = 0)$

$\boxed{6/10}$ $\Rightarrow Y_3$ and Y_4 are not independent

$\boxed{\text{Problem No 7}}$

(i) $f_{X_n}(x) = \binom{n}{x} p_n^x (1-p_n)^{n-x}$

$$= \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{x-1}{n})}{\lfloor x} (np_n)^x \left(1-\frac{np_n}{n}\right)^n (1-p_n)^{-x}$$

$I_{\{0,1,\ldots,n\}}(x)$, $n=1,2,\ldots$ ... $\boxed{\text{3 MARKS}}$

As $n\to\infty$, $\lim(np_n)=\lambda$; $\lim p_n = \lim \frac{1}{n} \lim np_n = 0$

$\Rightarrow f_{X_n}(x) \longrightarrow \frac{\lambda^x e^{-\lambda}}{\lfloor x} I_{\{0,1,2,\ldots\}}(x)$, as $n\to\infty$ ... $\boxed{\text{3 MARKS}}$

(ii) Let $X =$ # of wins in 5000 games. Then $X \sim Bin(5000, 0.001)$

$n = 5000$ is large; $p = 0.001$ is small and $np = 5 = \lambda$ (say)

By (i)

$P(X=x) \simeq \frac{e^{-\lambda}\lambda^x}{\lfloor x}$, $x = 0, 1, 2, \ldots, 5000$ ... $\boxed{\text{2 MARKS}}$

$P(X\geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - e^{-5}\left[1 + 5 + \frac{25}{2}\right] = 1 - \frac{37}{2}e^{-5}$ ... $\boxed{\text{2 MARKS}}$

$\boxed{\text{Problem No. 8}}$ (i) For real constants a and b

$$a\bar{X} + bY = \sum_{i=1}^{n} \left(\frac{a}{n} + b a_i \right) X_i = \text{linear combination of independent normally distd r.v.s}$$

$$\sim N_1(\cdot, \cdot) \quad (\text{I})$$

$$\Rightarrow (\bar{X}, Y) \sim N_2(\cdot, \cdot) \; \ldots\ldots$$

$$Cov(\bar{X}, Y) = Cov\left(\frac{1}{n} \sum_{i=1}^{n} X_i, \sum_{j=1}^{n} a_j x_j \right) = \frac{1}{n} \sum_{i=1}^{n} a_i = 0 \quad \ldots (\text{II})$$

(I) + (II) $\Rightarrow$ $\bar{X}$ and Y are independent .... $\boxed{\text{3 MARKS}}$

(ii) $\left.\begin{array}{l} \bar{X} \sim N(0, \frac{1}{n}) \\ Y \sim N(0, 1) \end{array}\right\}$ indept $\Rightarrow$ $\left.\begin{array}{l} \sqrt{n}\,\bar{X} \sim N(0,1) \\ Y^2 \sim \chi^2_1 \end{array}\right\}$ indept

$$\Rightarrow \frac{\sqrt{n}\,\bar{X}}{\sqrt{Y^2/1}} \sim t_1 \Rightarrow Y_1 = \frac{\sqrt{n}\,\bar{X}}{|Y|} \sim t_1 \quad \ldots \boxed{\text{4 MARKS}}$$

Thus pdf of Y_1 is $f_{Y_1}(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}, \; -\infty < y < \infty$

(iii) $\left.\begin{array}{l} \frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1) \\ \frac{X_3 + X_4}{\sqrt{2}} \sim N(0,1) \end{array}\right.$ $\Rightarrow$ $\frac{X_1 - X_2}{|X_3 + X_4|} \sim t_1 \Rightarrow Y_2 \stackrel{d}{=} |t_1|$ ... $\boxed{\text{3 MARKS}}$

The pdf of Y_2 is $f_{Y_2}(y) = \frac{2}{\pi} \cdot \frac{1}{1+y^2}, \; y > 0.$

Problem No. 9 (i) Let X be a rv s.t.

$$P(X = Var(X_i)) = p_i \quad i=1,2,\dots,n,$$

and let $g:[0,\infty)\to\mathbb{R}$ be defined by $g(x) = -\sqrt{x}$, $x \geq 0$. Then g is convex on $(0,\infty)$. By Jensen's inequality

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow \sum_{i=1}^{n}\left(-\sqrt{Var(X_i)}\right)p_i \geq -\sqrt{\sum_{i=1}^{n} Var(X_i)\,p_i}$$

$$\Rightarrow \sum_{i=1}^{n} p_i\sqrt{Var(X_i)} \leq \sqrt{\sum_{i=1}^{n} p_i\, Var(X_i)}$$

.... 3 MARKS

Also

$$Var\left(\sum_{i=1}^{n} p_i X_i\right) = \sum_{i=1}^{n} p_i^2\, Var(X_i) + 2\sum\sum_{1\leq i<j\leq n} p_i p_j\, Cov(X_i, X_j)$$

$$\leq \sum_{i=1}^{n} p_i^2\, Var(X_i) + 2\sum\sum_{1\leq i<j\leq n} p_i p_j \sqrt{Var(X_i)}\sqrt{Var(X_j)}$$

$$= \left(\sum_{i=1}^{n} p_i\sqrt{Var(X_i)}\right)^2$$

$$\Rightarrow \sqrt{Var\left(\sum_{i=1}^{n} p_i X_i\right)} \leq \sum_{i=1}^{n} p_i\sqrt{Var(X_i)} \leq \sqrt{\sum_{i=1}^{n} p_i\, Var(X_i)}$$

.... 2 MARKS

(ii) Let X be a rv such that

$$P(X = \sqrt{a_i}) = p_i \quad i=1,2,\dots,n$$

Let $g:[1,\infty)\to\mathbb{R}$ be defined by $g(x) = x^2(x^2-1)$, $x \geq 1$. Then g is convex on $(1,\infty)$.

$g''(x) = 12x^2 - 2 > 0 \quad \forall x \geq 1 \Rightarrow g$ is convex. .... 2 MARKS

Using Jensen's inequality

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow \sum_{i=1}^{n} a_i(a_i - 1)p_i \geq \left(\sum_{i=1}^{n}\sqrt{a_i}\,p_i\right)^2\left[\left(\sum_{i=1}^{n}\sqrt{a_i}\,p_i\right)^2 - 1\right]$$

.... 3 MARKS

$\boxed{\text{Problem No. 10}}$ (i) Let $Y_1 = Z_1$ and $Y_2 = 2Z_1 + 3Z_2$. Then

$(Y_1, Y_2) \sim N_2(0, 0, 1, 5, \rho)$, $\rho = \frac{2}{\sqrt{13}}$ ... $\boxed{\text{2 MARKS}}$

(since any linear combination of Y_1 and Y_2 = linear combination of Z_1 and Z_2 $\sim N(\cdot, \cdot)$)

$$\Rightarrow Y_2 | Y_1 = 1 \sim N\left(0 + \rho \frac{\sigma_2}{\sigma_1}(y_1 - \mu_1),\ \sigma_2^2(1-\rho^2)\right)$$

$$= N(2, 9) \quad \dots \quad \boxed{\text{3 MARKS}}$$

$$f_{Y_2|Y_1=1}(y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{18}(y_2-2)^2}, \quad -\infty < y_2 < \infty.$$

(ii) $$\max\{Z_1, Z_2\} = \frac{Z_1 + Z_2 + |Z_1 - Z_2|}{2}$$

$$E(\max\{Z_1, Z_2\}) = \frac{E(|Z_1 - Z_2|)}{2} \quad \dots \quad \boxed{\text{2 MARKS}}$$

$$Z_1 - Z_2 \sim N(0, 1+1-1) = N(0, 1)$$

$$\Rightarrow E[\max\{Z_1, Z_2\}] = \frac{1}{2} \int_{-\infty}^{\infty} |z| \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} z e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} dt$$

$$= \frac{1}{\sqrt{2\pi}}. \quad \dots \quad \boxed{\text{3 MARKS}}$$

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$\boxed{10/10}$